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Methods for Solving Some Type of Improper Fractional Integral

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Abstract: **In this paper, we use differentiation under fractional integral sign and integration by parts for fractional calculus to find the exact solution of some type of improper fractional integral. Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions play important roles in this paper. On the other hand, some examples are provided to illustrate our result. In fact, our result is a generalization of ordinary calculus result.**

Keywords: **Differentiation under fractional integral sign, integration by parts for fractional calculus, exact solution, improper fractional integral, Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions.**

I. INTRODUCTION

In mathematical analysis, a fractional derivative is a derivative of any arbitrary order, whether it is a real number or a complex number. The concept of fractional operators has been introduced almost simultaneously with the development of the classical calculus. The first known reference can be found in the letter of Leibniz and L'Hospital in 1695, which raised the question of the meaning of the semi-derivative. This problem has thus attracted the interest of a lot of famous mathematicians, including Euler, Liouville, Laplace, Riemann, Riesz, Grünwald, Letnikov, Weyl and many others. Since the 19th century, the theory of fractional calculus has developed rapidly, mainly as a foundation for a number of applied disciplines, including fractional differential equations and fractional dynamics. The applications of fractional calculus are very wide nowadays. Almost all modern engineering and science has been affected by the tools and techniques of fractional calculus. For example, it can find wide and fruitful applications in economics, viscoelasticity, physics, control theory, electrical engineering, biology, and so on [1-10].

However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-14]. Because Jumarie's modified R-L fractional derivative helps avoid non-zero fractional derivative of constant functions, it is easier to use this definition to associate fractional calculus with classical calculus.

In this paper, we use differentiation under fractional integral sign and integration by parts for fractional calculus to solve the following improper α-fractional integral:

$$\left({}_{0}I_{+\infty}^{\alpha}\right)\left[E_{\alpha}\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)\otimes_{\alpha} cos_{\alpha}(tx^{\alpha})\right], \tag{1}$$

where $0<\alpha\leq 1$ and t is a real number. Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions play important roles in this paper. And our result is a generalization of traditional calculus result.

II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([15])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \tag{2}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{3}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([16])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{0}D_x^{\alpha}\right)\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{0}D_x^{\alpha}\right)[C] = 0. \tag{5}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([17])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([18])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{7}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ & = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{9}$$

Definition 2.5 ([19])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the n-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha (-1)}$.

Definition 2.6 ([20])**:** Let $0 < \alpha \le 1$, and x be a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}. \tag{10}$$

Definition 2.7 ([21])**:** The α-fractional cosine and sine function are defined respectively as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n\alpha+1)} x^{2n\alpha} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{11}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma((2n+1)\alpha+1)} x^{(2n+1)\alpha} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{12}$$

Theorem 2.8 (differentiation under fractional integral sign) ([22])**:** *Assume that* $0 < \alpha \le 1$, t *is a nonzero real variable, and* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function at* $x = 0$, *then*

$$\frac{d}{dt}\left({}_0I_x^\alpha\right)[f_\alpha(tx^\alpha)] = \left({}_0I_x^\alpha\right)\left[\frac{d}{dt}f_\alpha(tx^\alpha)\right]. \tag{13}$$

Theorem 2.9 (integration by parts for fractional calculus) ([23])**:** *Suppose that* $0 < \alpha \le 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = \left[f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\right]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha)\otimes_\alpha \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{14}$$

Theorem 2.10 ([24])**:** *If* $0 < \alpha \le 1$, *and* $(-1)^\alpha$ *exists, then the* α*-fractional Gaussian integral*

$$\left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right] = \frac{\sqrt{\pi}}{2}. \tag{15}$$

III. MAIN RESULT

In this section, we will use differentiation under fractional integral sign and integration by parts for fractional calculus to find the solution of some type of improper fractional integral. Moreover, some examples are given to illustrate our result.

Theorem 3.1: Let $0 < \alpha \le 1$ and t be a real number. Then the improper α-fractional integral

$$\left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha(tx^\alpha)\right] = \frac{\sqrt{\pi}}{2}\cdot e^{-\frac{1}{4}t^2}. \tag{16}$$

Proof Let $F(t) = \left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha(tx^\alpha)\right]$, then by differentiation under fractional integral sign and integration by parts for fractional calculus,

$$\frac{d}{dt}F(t)$$

$$= \frac{d}{dt}\left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha(tx^\alpha)\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha \frac{d}{dt}cos_\alpha(tx^\alpha)\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[-\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha sin_\alpha(tx^\alpha)\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[sin_\alpha(tx^\alpha)\otimes_\alpha \left({}_0D_x^\alpha\right)\left[\frac{1}{2}E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]\right]$$

$$= \left[\frac{1}{2}E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha sin_\alpha(tx^\alpha)\right]_0^{+\infty} - \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{2}E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha \left({}_0D_x^\alpha\right)[sin_\alpha(tx^\alpha)]\right]$$

$$= -\frac{1}{2}t\cdot\left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha(tx^\alpha)\right]$$

$$= -\frac{1}{2}t\cdot F(t). \tag{17}$$

Therefore, $F(t)$ satisfies the first order ordinary differential equation

$$\frac{d}{dt}F(t)+\frac{1}{2}t\cdot F(t)=0. \tag{18}$$

Hence,

$$F(t)=C\cdot e^{-\frac{1}{4}t^2}. \tag{19}$$

Where C is a constant. By Theorem 2.10, we knows that

$$C=F(0)=\frac{\sqrt{\pi}}{2}. \tag{20}$$

Thus,

$$F(t)=\frac{\sqrt{\pi}}{2}\cdot e^{-\frac{1}{4}t^2}. \tag{21}$$

That is,

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha(tx^\alpha)\right]=\frac{\sqrt{\pi}}{2}\cdot e^{-\frac{1}{4}t^2}. \qquad \text{Q.e.d.}$$

Example 3.2: Let $0<\alpha\leq 1$. Then the improper α-fractional integrals

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha(5x^\alpha)\right]=\frac{\sqrt{\pi}}{2}\cdot e^{-\frac{25}{4}}, \tag{22}$$

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha(\sqrt{3}x^\alpha)\right]=\frac{\sqrt{\pi}}{2}\cdot e^{-\frac{3}{4}}, \tag{23}$$

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes_\alpha cos_\alpha\left(\frac{1}{2}x^\alpha\right)\right]=\frac{\sqrt{\pi}}{2}\cdot e^{-\frac{1}{16}}. \tag{24}$$

IV. CONCLUSION

In this paper, we use differentiation under fractional integral sign and integration by parts for fractional calculus to find the exact solution of some type of improper fractional integral. Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our result is a generalization of the result in classical calculus. In the future, we will continue to use our methods to study the problems in fractional differential equations and engineering mathematics.

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